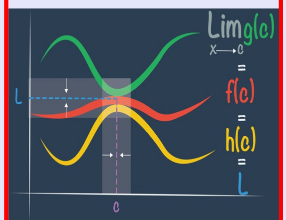
Math 261

Spring 2022

Lecture 5



$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

Evaluate $\displaystyle \lim_{h\to 0} \frac{\sin h}{h^2+2h} = \lim_{h\to 0} \frac{\sin h}{h(h+2)}$

$$= \lim_{h\to 0} \left[\frac{\sin h}{h} \cdot \frac{1}{h+2}\right]$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{h+2}$$

$$= 1 \cdot \frac{1}{0+2} = 1 \cdot \frac{1}{2} = \boxed{\frac{1}{2}}$$

Evaluate $\lim\limits_{x\to 2} \dfrac{\sin(x-2)}{x^2-3x+2} = \lim\limits_{x\to 2} \dfrac{\sin(x-2)}{(x-2)(x-1)}$

Recall

$\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$

$h = x-2$

$x\to 2,\ h\to 0$

$$= \lim_{x\to 2}\left[\frac{\sin(x-2)}{x-2}\cdot\frac{1}{x-1}\right]$$

$$= \lim_{x\to 2}\frac{\sin(x-2)}{x-2}\cdot\lim_{x\to 2}\frac{1}{x-1}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{x\to 2}\frac{1}{x-1}$$

$$= 1\cdot\frac{1}{2-1} = 1\cdot 1 = \boxed{1}$$

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

Recall

$\boxed{\lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1}$

$$= \lim_{x\to 0}\frac{\dfrac{3\sin 3x}{3x}}{\dfrac{5\sin 5x}{5x}}$$

$$= \frac{3}{5}\lim_{x\to 0}\frac{\dfrac{\sin 3x}{3x}}{\dfrac{\sin 5x}{5x}} = \frac{3}{5}\cdot\frac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}}$$

$$= \frac{3}{5}\cdot\frac{1}{1} = \boxed{\frac{3}{5}}$$

Show $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$ $\quad \frac{\sin^2 h}{1-\cos^2 h}$

$$\lim_{h\to 0}\frac{1-\cos h}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}\cdot\frac{1+\cos h}{1+\cos h}\right]$$

$\lim_{h\to 0}\frac{\sin h}{h} = 1$

$$= \lim_{h\to 0}\frac{\sin^2 h}{h(1+\cos h)}$$

$$= \lim_{h\to 0}\frac{\sin h\cdot\sin h}{h(1+\cos h)}$$

$$= \boxed{\lim_{h\to 0}\frac{\sin h}{h}}\cdot\lim_{h\to 0}\frac{\sin h}{1+\cos h}$$

$$= 1 \cdot \frac{\sin 0}{1+\cos 0}$$

$$= 1\cdot\frac{0}{1+1} = 1\cdot\frac{0}{2}$$

$$= 1\cdot 0 = \boxed{0}$$

Now

$$\boxed{\lim_{h\to 0}\frac{\sin h}{h} = 1}$$

$$\boxed{\lim_{h\to 0}\frac{1-\cos h}{h} = 0}$$

Evaluate $\lim_{x\to 0}\frac{\tan 7x}{\sin 3x}$

$\frac{\sin 7x}{\cos 7x} \div \sin 3x$

$= \frac{\sin 7x}{\cos 7x}\cdot\frac{1}{\sin 3x}$

$$= \lim_{x\to 0}\frac{\frac{\sin 7x}{\cos 7x}}{\sin 3x} = \lim_{x\to 0}\frac{\sin 7x}{\cos 7x\,\sin 3x}$$

$$= \lim_{x\to 0}\left[\frac{1}{\cos 7x}\cdot\frac{\sin 7x}{\sin 3x}\right]$$

$$= \boxed{\lim_{x\to 0}\frac{1}{\cos 7x}}\cdot\lim_{x\to 0}\frac{7\frac{\sin 7x}{7x}}{3\frac{\sin 3x}{3x}}$$

$$= \frac{1}{\cos(7\cdot 0)}\cdot\frac{7}{3}\cdot\frac{\lim_{x\to 0}\frac{\sin 7x}{7x}}{\lim_{x\to 0}\frac{\sin 3x}{3x}}$$

$$= \frac{1}{1}\cdot\frac{7}{3}\cdot\frac{1}{1} = \boxed{\frac{7}{3}}$$

Evaluate $\lim_{x\to0}\frac{x^2}{1-\cos x} = \frac{0^2}{1-\cos 0} = \frac{0}{1-1} = \frac{0}{0}$

I.F.

$$\lim_{x\to0}\frac{x^2}{1-\cos x} = \lim_{x\to0}\left[\frac{x^2}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\right]$$

$$= \lim_{x\to0}\left[\frac{x^2}{\sin^2 x}\cdot(1+\cos x)\right]$$

$$= \lim_{x\to0}\frac{x^2}{\sin^2 x}\cdot\lim_{x\to0}(1+\cos x)$$

$$= \left[\lim_{x\to0}\frac{x}{\sin x}\right]^2\cdot(1+\cos 0)$$

$$= 1^2\cdot(1+1) = \boxed{2}$$

Evaluate $\lim_{x\to0}\frac{1-\cos 5x}{1-\cos 8x} = \frac{1-\cos(5\cdot0)}{1-\cos(8\cdot0)} = \frac{1-\cos 0}{1-\cos 0} = \frac{1-1}{1-1} = \frac{0}{0}$

I.F.

$$\lim_{x\to0}\frac{1-\cos 5x}{1-\cos 8x} = \lim_{x\to0}\frac{\frac{5(1-\cos 5x)}{5x}}{\frac{8(1-\cos 8x)}{8(x)}}$$

$$= \frac{5}{8}\lim_{x\to0}\frac{\frac{1-\cos 5x}{5x}}{\frac{1-\cos 8x}{8x}} = \frac{5}{8}\cdot\frac{0}{0} = \frac{0}{0}$$

$$\lim_{x\to0}\frac{1-\cos 5x}{1-\cos 8x} = \lim_{x\to0}\left[\frac{1-\cos 5x}{1-\cos 8x}\cdot\frac{1+\cos 5x}{1+\cos 5x}\cdot\frac{1+\cos 8x}{1+\cos 8x}\right]$$

$$= \lim_{x\to0}\left[\frac{\sin^2 5x}{\sin^2 8x}\cdot\frac{1+\cos 8x}{1+\cos 5x}\right] = \lim_{x\to0}\frac{1+\cos 8x}{1+\cos 5x}\cdot\lim\left[\frac{\sin 5x}{\sin 8x}\right]^2$$

$$= \frac{1+1}{1+1}\xrightarrow{}1\cdot\left[\lim_{x\to0}\frac{\frac{5\sin 5x}{5x}}{\frac{8\sin 8x}{8x}}\right]^2 = \left[\frac{5}{8}\lim_{x\to0}\frac{\frac{\sin 5x}{5x}}{\frac{\sin 8x}{8x}}\right]^2 = \left[\frac{5}{8}\cdot\frac{1}{1}\right]^2$$

$$= \boxed{\frac{25}{64}}$$

Find a nonzero value for constant K such that

$$f(x)=\begin{cases}\dfrac{\tan Kx}{x} & \text{if } x<0\\[2mm] 3x+2K^2 & \text{if } x\geq 0\end{cases}$$

is continuous at $x=0$.

1) find $f(0)$

$f(0)=3(0)+2K^2$

$\boxed{f(0)=2K^2}$

$$\lim_{x\to 0} f(x)=f(0)$$

$$\lim_{x\to 0} f(x) \Leftrightarrow \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{\tan Kx}{x}=\lim_{x\to 0^-}\frac{\sin Kx}{\cos Kx}\cdot\frac{1}{x}$$

$$=\lim_{x\to 0^-}\left[\frac{\sin Kx}{x}\cdot\frac{1}{\cos Kx}\right]$$

$$=\lim_{x\to 0^-}\frac{K\sin Kx}{Kx}\cdot\lim_{x\to 0^-}\frac{1}{\cos Kx}$$

$$=K\boxed{\lim_{x\to 0}\frac{\sin Kx}{Kx}}\cdot\frac{1}{\cos 0}$$

$$=K\cdot\boxed{1}\cdot\frac{1}{1}=K$$

$\lim_{x\to 0^+} f(x)=$

$\lim_{x\to 0^+}(3x+2K^2)=\boxed{2K^2}$

$2K^2=K$

$2K^2-K=0$

$K(2K-1)=0$

$K=0$ $\quad$ $2K-1=0 \to K=\frac{1}{2}$

Final Ans $K=\frac{1}{2}$

For $\varepsilon=.1$ find a $\delta>0$ such that $\lim_{x\to 2}\frac{x}{2}=1$

$\lim_{x\to 2}\frac{x}{2}=1$

Verify $\lim_{x\to 2}\frac{x}{2}=\frac{2}{2}=1\checkmark$

$f(x)=\frac{x}{2}$

$a=2$

$L=1$

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|\frac{x}{2}-1\right|<\varepsilon$ " $|x-2|<\delta$

$\left|\frac{1}{2}(x-2)\right|<\varepsilon$ " $|x-2|<\delta$

$\frac{1}{2}|x-2|<\varepsilon$ " $|x-2|<\delta$

$|x-2|<2\varepsilon$ " $|x-2|<\delta$

Pick $\delta=2\varepsilon$

For $\varepsilon=.1$, Pick $\delta=2(.1)=\boxed{.2}$

For $\varepsilon = .05$, find a $\delta > 0$ such that

$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4.$

$f(x) = \frac{x^2-4}{x-2}$

$L = 4$, $a = 2$

Verify the limit $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2}$

$= \lim_{x \to 2} (x+2) = \boxed{4}$ ✓

For $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$\left|\frac{x^2-4}{x-2} - 4\right| < \varepsilon$ " $|x-2| < \delta$

$\left|\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} - 4\right| < \varepsilon$ " $|x-2| < \delta$

$|x+2-4| < \varepsilon$ " $|x-2| < \delta$

$|x-2| < \varepsilon$ " $|x-2| < \delta$

Pick $\delta = \varepsilon$ For $\varepsilon = .05$, we pick $\boxed{\delta = .05}$

For $\varepsilon = .05$, find a $\delta > 0$ such that $\lim_{x \to 5} \frac{1}{x} = \frac{1}{5}$

$\lim_{x \to 5} \frac{1}{x} = \frac{1}{5}$ ✓

$f(x) = \frac{1}{x}$

$a = 5$

$L = \frac{1}{5}$

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$\left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon$ " $|x-5| < \delta$

$\left|\frac{5-x}{5x}\right| < \varepsilon$ " $|x-5| < \delta$

$|5-x| = |x-5|$

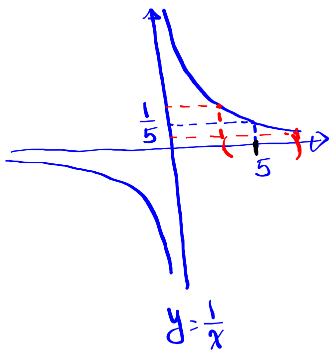
$\left|\frac{x-5}{5x}\right| < \varepsilon$ " $|x-5| < \delta$

$\frac{|x-5|}{|5x|} < \varepsilon$ " $|x-5| < \delta$

Keep $\frac{|x-5|}{5|x|} < \varepsilon$
Bound

For polynomial functions → we want $\delta < 1$

For rational functions → we need to be mindful of the domain.



$|x-5|<1$ — $\delta<1$

$-1<x-5<1$

$4<x<6$

$-6<x<6$

$|x|<6$

$5|x|<5\cdot 6$

$5|x|<30$

$\frac{1}{5|x|}>\frac{1}{30}$

$\frac{|x-5|}{5|x|}=\frac{|x-5|}{30}<\varepsilon$

$|x-5|<30\varepsilon$

$\delta=1$

For $\varepsilon=.05$, $\delta=30(.05)=1.5$

Pick $\delta=\min\{1, 30\varepsilon\}$

$\delta=\min\{1, 1.5\}$

For $\varepsilon=.05$

$\boxed{\text{Pick } \delta=1}$

Prove $\lim\limits_{x\to\frac{1}{2}}\frac{1}{x}=2$ ✓

$f(x)=\frac{1}{x}$

$L=2$

$a=\frac{1}{2}$

For any $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|\frac{1}{x}-2\right|<\varepsilon$ 〃 $|x-\frac{1}{2}|<\delta$

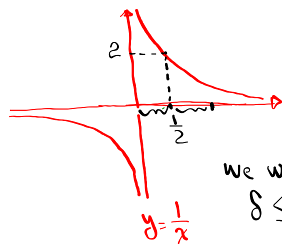
$\left|\frac{1-2x}{x}\right|<\varepsilon$ 〃 $|x-\frac{1}{2}|<\delta$

$\left|\frac{-2x+1}{x}\right|<\varepsilon$

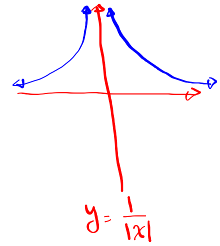
$\left|\frac{-2(x-\frac{1}{2})}{x}\right|<\varepsilon$

$\left|\frac{-2}{x}(x-\frac{1}{2})\right|<\varepsilon$

$\underbrace{\left|\frac{2}{x}\right|}_{\text{Bound}}\underbrace{\left|x-\frac{1}{2}\right|}_{\text{Keep}}<\varepsilon$



we want
$\delta \le \frac{1}{4}$



$x \neq 0$

$|x-\frac{1}{2}|<\frac{1}{4}$

$-\frac{1}{4}<x-\frac{1}{2}<\frac{1}{4}$

$-1<4x-2<1$

$1<4x<3$

$\frac{1}{4}<x<\frac{3}{4}$

$4>\frac{1}{x}>\frac{4}{3}$

$\frac{4}{3}<\frac{1}{x}<4$

$\color{red}{-8<}\frac{8}{3}<\frac{2}{x}<8$

$\left|\frac{2}{x}\right|<8$

$\left|\frac{2}{x}\right|\,|x-\frac{1}{2}|<\varepsilon$

$8\,|x-\frac{1}{2}|<\varepsilon$

$|x-\frac{1}{2}|<\frac{\varepsilon}{8}$

Pick $\delta=\min\left\{\frac{1}{4},\frac{\varepsilon}{8}\right\}$

Pick δ when $\varepsilon=4$

$\delta=\min\left\{\frac{1}{4},\frac{4}{8}\right\}$
$=\min\left\{\frac{1}{4},\frac{1}{2}\right\}$
$\boxed{\delta=\frac{1}{4}}$

Pick δ when $\varepsilon=.5$

$\delta=\min\left\{\frac{1}{4},\frac{.5}{8}\right\}=\min\left\{\frac{1}{4},\frac{1}{16}\right\}$ $\boxed{\delta=\frac{1}{16}}$

Prove $\lim\limits_{x\to 6}\sqrt{x+3}=3$ ✓

$f(x)=\sqrt{x+3}$
$a=6$
$L=3$

For $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|\sqrt{x+3}-3|<\varepsilon$, $|x-6|<\delta$

Using Conjugates

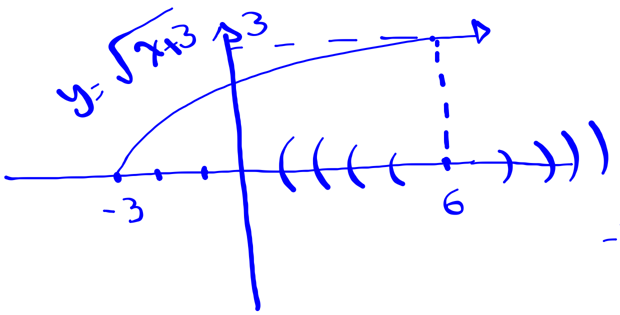
$\left|\left(\sqrt{x+3}-3\right)\cdot\frac{\sqrt{x+3}+3}{\sqrt{x+3}+3}\right|<\varepsilon$, $|x-6|<\delta$

$\left|\frac{x+3-9}{\sqrt{x+3}+3}\right|<\varepsilon$

$\left|\frac{x-6}{\sqrt{x+3}+3}\right|<\varepsilon$

Keep

Bound



Pick $\delta \le 1$

$|x-6|<1$

$-1< x-6<1$

$5< x<7$

$8< x+3<10$

$\sqrt{8}<\sqrt{x+3}<\sqrt{10}$

$\sqrt{8}+3<\sqrt{x+3}+3<\underbrace{\sqrt{10}+3}_{6.2}$

It is safe to say

$|\sqrt{x+3}+3|<7$

what about

$$\frac{1}{|\sqrt{x+3}+3|}>\frac{1}{7}$$

$\delta=\min\left\{1,\frac{1}{7}\varepsilon\right\}$